The Effect of Phase-Difference on the Spreading Rate of a Jet

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The role of initial phase-difference between fundamental and subharmonic instability waves on the spreading rate of a circular jet under bimodal excitation is investigated theoretically. For all Strouhal numbers considered, the initial growth rate of the subharmonic was found to be a maximum when the two waves are initially in-phase. The effect of phase-difference on the subharmonic's peak is dependent on Strouhal number. Development of the momentum thickness along the jet is characterized by regions of stepwise growth. The momentum thickness first increases as the fundamental amplifies. This growth region is followed by a region where the decay of the fundamental and the growth of the subharmonic result in a constant momentum thickness. Once the fundamental has fully decayed, the momentum thickness grows again as a result of the subharmonic's amplification. The growth rate is dependent on the initial phase-difference, and is maximum when the subharmonic suffers maximum amplification.

Introduction

NUMERICAL simulation of vortex-pairing interactions in two-dimensional mixing layers[1,2] has shown that the pairing process is dependent on the phase-difference between the fundamental and subharmonic instability waves. The smaller the initial phase-difference between the two modes, the faster the coalescence of the two vortices. In the limiting case where the two components are antiphase, the numerical results of Patnaik et al.[1] showed that the vortex-pairing interaction is actually suppressed and replaced by shredding interactions. In the experiment of Zhang et al.[3] of a two-dimensional shear layer under bimodal excitation, significantly different merging patterns were observed as a result of changing the initial phase-difference between the fundamental and subharmonic. Monkewitz[4] modified Kelly's[5] temporal stability analysis of a spatially periodic mixing layer to include an arbitrary phase-difference between the fundamental and subharmonic instability waves. The initial growth rate was found to vary continuously from maximum when both waves are in-phase, to a minimum when they are out of phase. These investigations indicate the importance of the phase-difference on the fundamental-subharmonic interaction.

In the plane shear-layer case, Ho and Huang[6] have shown that the spreading rates can be greatly manipulated by controlling the vortex-pairing process. Since this pairing process is dependent on the phase-difference between the fundamental and subharmonic instabilities,[1-5] one would expect the spreading rate to be dependent on this phase-difference. The present work is concerned with the technologically important problem of a round jet where the spreading rate can be manipulated via bimodal excitation at fundamental and subharmonic frequencies. In the present analysis, the energy equations for each flow component are used to study the effect of initial phase-differencing between the two excitation components on their spatial developments and on the spreading rate. In the jet mixing region, the vortex-pairing process results in the growth of the subharmonic. The location of pairing and, thus, of the subharmonic's amplification depends on the random initial fluctuations at the jet exit and, hence, on the initial phase-difference. The question arises as to how much energy is carried by the subharmonic and by the fundamental. The integral technique used here is best suited for that purpose.

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Problem Formulation

Consider a circular jet with coordinates (x,r,θ) in the axial, radial, and azimuthal directions, respectively. (u,v,w) are the corresponding velocity components. All velocities are normalized by the jet exit velocity u_e, and all distances are normalized by the jet exit radius R. Pressure is normalized by the dynamic head. For an axisymmetric jet, the continuity and Navier-Stokes equations, in cylindrical coordinates, can be written as

$$\frac{\partial u}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(rv)=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}=-\frac{\partial p}{\partial x}+\frac{1}{Re}[\nabla^2 u]$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial r}=-\frac{\partial p}{\partial r}+\frac{1}{Re}[\nabla^2 v-v/r^2]$$

where

$$\nabla^2=\frac{\partial}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) \tag{1}$$

Re is the Reynolds number, where $Re=u_eR/\nu$ and ν is the kinematic viscosity. Now, each flow component is divided into a mean part independent of time and a fluctuating part with zero mean, i.e.,

$$u(x,r,t)=\bar{u}(x,r)+\tilde{u}(x,r,t)$$

$$v(x,r,t)=\bar{v}(x,r)+\tilde{v}(x,r,t)$$

$$p(x,r,t)=\bar{p}(x,r)+\tilde{p}(x,r,t) \tag{2}$$

The fluctuations $\tilde{u}$, $\tilde{v}$, and $\tilde{p}$ are assumed to be periodic in time and, therefore, their time average is zero. The time dependence of the fluctuating quantities is assumed to be in the form $(e^{im\omega t}+\text{c.c.})$. Here, ω is the frequency and is taken to be real, m is an integer, and c.c. or an asterisk denote a complex conjugate. Substituting Eq. (2) into Eq. (1) and averaging over time, one obtains the mean-flow momentum and continuity equations:

$$\frac{\partial \bar{u}}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(r\bar{v})=0$$

$$\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial r}+\frac{\partial}{\partial x}(\overline{\tilde{u}\tilde{u}})+\frac{1}{r}\frac{\partial}{\partial r}(\overline{r\tilde{u}\tilde{v}})$$

$$=-\frac{\partial \bar{p}}{\partial x}+\frac{1}{Re}[\nabla^2\bar{u}]$$

$$\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial r}+\frac{\partial}{\partial x}(\overline{\tilde{v}\tilde{u}})+\frac{1}{r}\frac{\partial}{\partial r}(\overline{r\tilde{u}\tilde{v}})$$

$$=-\frac{\partial \bar{p}}{\partial r}+\frac{1}{Re}[\nabla^2\bar{v}-\bar{v}/r^2] \tag{3}$$

Subtracting Eq. (3) from Eq. (1), the continuity and momentum equations of the fluctuations are obtained in the form:

$$\frac{\partial \tilde{u}}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{v})=0$$

$$\frac{\partial \tilde{u}}{\partial t}+\bar{u}\frac{\partial \tilde{u}}{\partial x}+\bar{v}\frac{\partial \tilde{u}}{\partial r}+\tilde{u}\frac{\partial \bar{u}}{\partial x}+\tilde{v}\frac{\partial \bar{u}}{\partial r}+\frac{\partial}{\partial x}[\tilde{u}\tilde{u}-\overline{\tilde{u}\tilde{u}}]$$

$$+\frac{1}{r}\frac{\partial}{\partial r}[r(\tilde{u}\tilde{v}-\overline{\tilde{u}\tilde{v}})]=-\frac{\partial \tilde{p}}{\partial x}+\frac{1}{Re}[\nabla^2\tilde{u}]$$

$$\frac{\partial \tilde{v}}{\partial t}+\bar{u}\frac{\partial \tilde{v}}{\partial x}+\bar{v}\frac{\partial \tilde{v}}{\partial r}+\tilde{u}\frac{\partial \bar{v}}{\partial x}+\tilde{v}\frac{\partial \bar{v}}{\partial r}+\frac{\partial}{\partial x}[\tilde{u}\tilde{v}-\overline{\tilde{u}\tilde{v}}]$$

$$+\frac{1}{r}\frac{\partial}{\partial r}[r(\tilde{v}\tilde{v}-\overline{\tilde{v}\tilde{v}})]=-\frac{\partial \tilde{p}}{\partial x}+\frac{1}{Re}[\nabla^2\tilde{v}-\tilde{v}/r^2] \tag{4}$$

The energy equation for the mean flow is obtained by multiplying each mean-flow momentum equation by the corresponding velocity and adding

$$\frac{\partial}{\partial x}[\bar{u}(\bar{u}^2+\bar{v}^2)/2]+\frac{1}{r}\frac{\partial}{\partial r}[r\bar{v}(\bar{u}^2+\bar{v}^2)/2]$$

$$=-\frac{\partial(\bar{p}\bar{u})}{\partial x}-\frac{1}{r}\frac{\partial(\bar{p}\bar{v})}{\partial r}$$

$$-\left[-\overline{\tilde{u}\tilde{u}}\frac{\partial \bar{u}}{\partial x}-\overline{\tilde{u}\tilde{v}}\frac{\partial \bar{u}}{\partial r}-\overline{\tilde{u}\tilde{v}}\frac{\partial \bar{v}}{\partial x}-\overline{\tilde{v}\tilde{v}}\frac{\partial \bar{v}}{\partial r}\right]$$

$$-\frac{\partial}{\partial x}[\overline{\tilde{u}\tilde{u}}\bar{u}+\overline{\tilde{u}\tilde{v}}\bar{v}]-\frac{1}{r}\frac{\partial}{\partial r}[r(\overline{\tilde{u}\tilde{v}}\bar{u}+\overline{\tilde{v}\tilde{v}}\bar{v})]$$

$$+\frac{1}{Re}[\bar{u}\nabla^2\bar{u}+\bar{v}\nabla^2\bar{v}-\bar{v}^2/r^2] \tag{5}$$

Now, each fluctuating quantity is decomposed into two fluctuation components: u_i' and u_i'', where $u_i' \sim (e^{i\omega t}+\text{c.c.})$ and $u_i'' \sim (e^{2i\omega t}+\text{c.c.})$. Thus,

$$\tilde{u}=u'+u'', \quad \tilde{v}=v'+v'', \quad \tilde{p}=p'+p''$$

Hence, u_i'' and p'' are the fundamental's components and u_i' and p' are the subharmonic components. The energy equations for the subharmonic and the fundamental are obtained by multiplying Eq. (4) by u_i' and u_i'', respectively, and taking the time average. Thus, for the subharmonic, we obtain:

$$\frac{\partial}{\partial x}[\bar{u}\overline{Q'}]+\frac{1}{r}\frac{\partial}{\partial r}[r\bar{v}\overline{Q'}]=-\frac{\partial}{\partial x}(\overline{p'u'})$$

$$-\frac{1}{r}\frac{\partial}{\partial r}(r\overline{p'v'})=-\overline{u'u'}\frac{\partial \bar{u}}{\partial x}-\overline{u'v'}\frac{\partial \bar{u}}{\partial r}$$

$$-\overline{u'v'}\frac{\partial \bar{v}}{\partial x}-\overline{v'v'}\frac{\partial \bar{v}}{\partial r}-\left[\overline{u'\frac{\partial}{\partial x}\tilde{u}\tilde{u}}\right.$$

$$\left.+\overline{u'\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{u}\tilde{v})}+\overline{v'\frac{\partial}{\partial x}\tilde{u}\tilde{v}}+\overline{v'\frac{1}{r}\frac{\partial}{\partial r}(rvv)}\right]$$

$$+\frac{1}{Re}[\overline{u'\nabla^2u'}+\overline{v'\nabla^2v'}-\bar{v}'^2/r^2]$$

where

$$Q=(u^2+v^2)/2 \tag{6}$$

And a similar equation is obtained for the fundamental. For high Reynolds numbers, the boundary-layer-type approximations can be applied to mean quantities. These approximations imply $\bar{v} \ll \bar{u}$, $\partial(\bar{\ })/\partial x \ll \partial(\bar{\ })/\partial r$. Applying these approximations to the energy equations and integrating over r, one obtains the following energy equations for the mean flow, the fundamental, and the subharmonic, respectively:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty u^3/2r\mathrm{d}r=-\int_0^\infty -\overline{u'v'}\frac{\partial \bar{u}}{\partial r}r\mathrm{d}r$$

$$-\int_0^\infty -\overline{u''v''}\frac{\partial \bar{u}}{\partial r}r\mathrm{d}r-\frac{1}{Re}\int_0^\infty\left(\frac{\partial \bar{u}}{\partial r}\right)^2 r\mathrm{d}r \tag{7a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \bar{u}\overline{Q''}r\mathrm{d}r=\int_0^\infty -\overline{u''v''}\frac{\partial \bar{u}}{\partial r}r\mathrm{d}r$$

$$-\int_0^\infty\left[\overline{\tilde{u}\tilde{u}\frac{\partial u''}{\partial x}}+\overline{\tilde{u}\tilde{v}\left(\frac{\partial u''}{\partial r}+\frac{\partial v''}{\partial r}\right)}+\overline{\tilde{v}\tilde{v}\frac{\partial v''}{\partial r}}\right]$$

$$-\frac{1}{Re}\int_0^\infty\left[\left(\frac{\partial u''}{\partial x}\right)^2+\left(\frac{\partial u''}{\partial r}\right)^2+\left(\frac{\partial v''}{\partial x}\right)^2\right.$$

$$\left.+\left(\frac{\partial v''}{\partial r}\right)^2-\left(\frac{v''}{r^2}\right)^2\right]r\mathrm{d}r \tag{7b}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \bar{u}\overline{Q'}r\mathrm{d}r=\int_0^\infty -\overline{u'v'}\frac{\partial \bar{u}}{\partial r}r\mathrm{d}r$$

$$-\int_0^\infty\left[\overline{u'u'}\frac{\partial u'}{\partial x}+\overline{\tilde{u}\tilde{v}\left(\frac{\partial u'}{\partial r}+\frac{\partial v'}{\partial x}\right)}+\overline{\tilde{v}\tilde{v}\frac{\partial v'}{\partial r}}\right]r\mathrm{d}r$$

$$-\frac{1}{Re}\int_0^\infty\left[\overline{\left(\frac{\partial u'}{\partial x}\right)^2}-\frac{v'^2}{r^2}\right.$$

$$\left.+\overline{\left(\frac{\partial u'}{\partial r}\right)^2+\left(\frac{\partial v'}{\partial x}\right)^2+\left(\frac{\partial v'}{\partial r}\right)^2}\right] \tag{7c}$$

This set of equations (7) represents the energy exchanges between the three flow components. In the integral energy approach, the unknowns may be approximated by few shape parameters which, in turn, will be determined by the integral equations. The mean-flow shape is taken here as the two-stage hyperbolic tangent profile introduced by Michalke,[7] which describes the flow in the initial region of the jet. The mean velocity profile is given in terms of r and $\theta(x)$, where θ is the momentum thickness to be determined from the nonlinear analysis. As in Refs. 8 and 9, the shape functions for the fundamental and subharmonic components are determined via the local linear stability analysis. The subharmonic

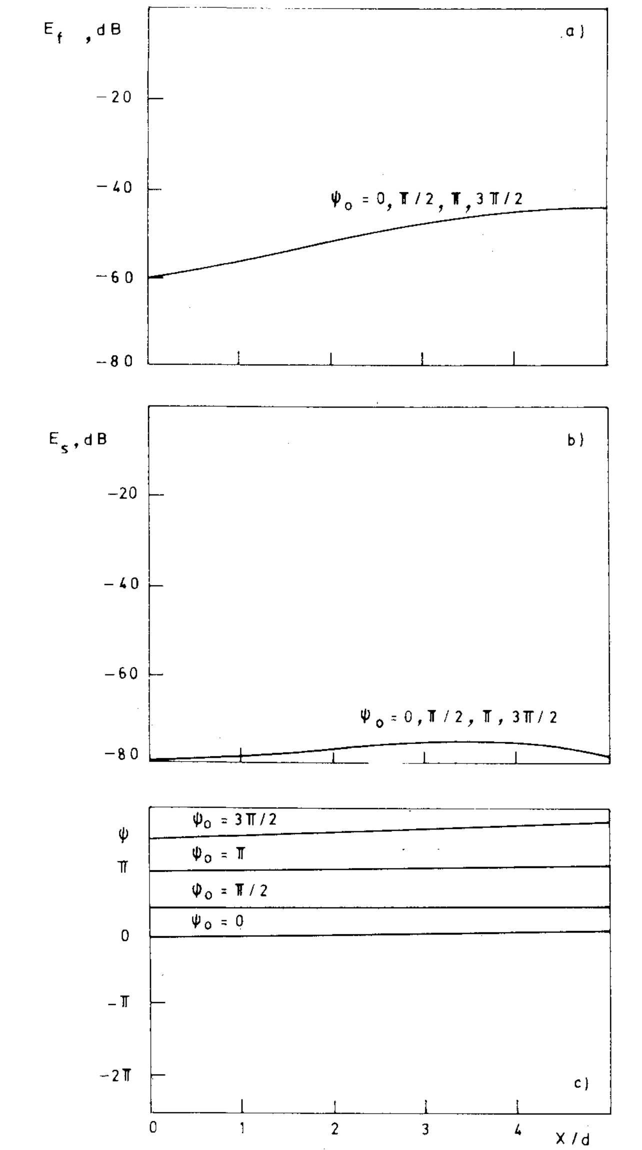

Fig. 1 Effect of ψ_0 on the energies of the fundamental and subharmonic and on their effective phase-difference at $S_f = 0.2$: a) fundamental's energy; b) subharmonic's energy; c) effective phase-difference.

and fundamental components are assumed to be in the form:

$$[u',v',p'] = F'(x)[\hat{u}'(r),\hat{v}'(r),\hat{p}'(r)]\exp(-i\omega t) + \text{c.c.}$$

$$[u'',v'',p''] = F''(x)[\hat{u}''(r),\hat{v}''(r),\hat{p}''(r)] \times \exp(-2i\omega t + i\psi_0) + \text{c.c.} \tag{8}$$

where ψ_0 is a prescribed initial phase-difference between the fundamental and the subharmonic. F' and F'' are the amplitude functions which, to a first approximation, are given by the local linear stability theory in which $\mathrm{d}F(x)/\mathrm{d}x = i\alpha F(x)$, where α is the complex wave number corresponding to the frequency. Subsequently, $F(x)$ will be determined by the nonlinear analysis. $\hat{u}(r)$, $\hat{v}(r)$, and $\hat{p}(r)$ are obtained from linear stability theory, where the nonlinear terms are neglected in Eq. (4) and the Reynolds number is assumed to be large enough for the inviscid solution to be a good approximation. The solution of the linear stability problem follows Michalke's[7] formulation for the amplified solution. Beyond the neutral point, the damped solution is obtained by taking the problem of the complex r plane and following a rectangular contour of integration, as in Ref. 10.

Since the mean velocity profile is a function of $\theta(x)$, the linear stability solution is obtained for several downstream stations. Thus, $\hat{u}$, $\hat{v}$, $\hat{p}$, and α are obtained as the eigenfunctions and eigenvalue corresponding to a given frequency at a certain axial location (x). These eigenfunctions are normalized such that $[F(x)]^2$ represents the energy integrated across the jet. Substituting these shape functions in Eqs. (7a–7c), the energy equations for the mean flow, the fundamental, and the subharmonic, respectively, reduce to

$$I_{cm}\frac{\mathrm{d}\theta}{\mathrm{d}x} = -I_{pf}E_f - I_{ps}E_s - I_{dm}/Re \tag{9a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[I_{cf}E_f] = I_{pf}E_f - I_{fs}E_s\sqrt{E_f} - I_{df}E_f/Re \tag{9b}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[I_{cs}E_s] = I_{ps}E_s + I_{fs}E_s\sqrt{E_f} - I_{ds}E_s/Re \tag{9c}$$

where

$$E_f(x) = |F''(x)|^2, \quad E_s(x) = |F'(x)|^2$$

$$I_{cm}(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\int_0^\infty u^3/2r\mathrm{d}r$$

$$I_{cf}(\theta,S_f) = 0.5\int_0^\infty \bar{u}(|\hat{u}''|^2 + |\hat{v}''|^2)r\mathrm{d}r$$

$$I_{cs}(\theta,S_s) = 0.5\int_0^\infty u(|\hat{u}'|^2 + |\hat{v}'|^2)r\mathrm{d}r$$

$$I_{pf}(\theta,S_f) = \int_0^\infty -(\hat{u}''\hat{v}'' + \hat{u}''^*\hat{v}''^*)r\mathrm{d}r$$

$$I_{ps}(\theta,S_s) = \int_0^\infty -(\hat{u}'\hat{v}' + \hat{u}'^*\hat{v}'^*)r\mathrm{d}r$$

$$I_{fs}(\theta,S_f,S_s) = \exp(2i\psi_s - i\psi_f - i\psi_0)I_w + \text{c.c.}$$

$$I_w(\theta,S_f,S_s) = -\int_0^\infty\left[\hat{u}'\hat{u}'\frac{\partial\hat{u}''^*}{\partial x} + \hat{u}'\hat{v}'\left(\frac{\partial\hat{u}''^*}{\partial r} + \frac{\partial\hat{v}''^*}{\partial x}\right) + \hat{v}'\hat{v}'\frac{\partial v''^*}{\partial r}\right]r\mathrm{d}r + \text{c.c.}$$

$$I_{dm}(\theta) = \int_0^\infty\left(\frac{\partial\bar{u}}{\partial r}\right)r\mathrm{d}r$$

$$I_{df}(\theta,S_f) = \int_0^\infty\left[\left|\frac{\partial\hat{u}''}{\partial r}\right|^2 + \left|\frac{\partial\hat{v}''}{\partial r}\right|^2 + \left|\frac{\partial\hat{u}''}{\partial r}\right|^2 + \left|\frac{\partial\hat{v}''}{\partial r}\right|^2 - |\hat{v}''|^2/r^2\right]r\mathrm{d}r$$

$$I_{ds}(\theta,S_s) = \int_0^\infty\left[\left|\frac{\partial\hat{u}'}{\partial r}\right|^2 + \left|\frac{\partial\hat{v}'}{\partial r}\right|^2 + \left|\frac{\partial\hat{u}'}{\partial x}\right|^2 + \left|\frac{\partial\hat{v}'}{\partial r}\right|^2 - |v'|^2/r^2\right]r\mathrm{d}r$$

S is the Strouhal number defined as fd/u_e, where d is the nozzle diameter and f is the frequency in Hz ($f = \omega/2\pi$). ψ_f and ψ_s are the phase angles of $F''(x)$ and $F'(x)$, respectively. The first and second terms in Eq. (9a) represent the productions of the fundamental and subharmonic, respectively, by the mean flow. The second term in Eqs. (9b) and (9c) is the energy exchange between the fundamental and the subharmonic. This fundamental-subharmonic interaction is dependent on the effective phase-difference. $\psi = 2\psi_s - \psi_f - \psi_0$. The last terms in Eqs. (9a-9c) are the viscous dissipation by the mean flow, the fundamental, and the subharmonic, respectively. Nikotopoulos and Liu[11] have indicated that for Reynolds numbers greater than 100 these viscous dissipations are negligible. Since attention is focused here on the high-Reynolds-number case, these viscous terms will be neglected in the subsequent calculations. However, for Reynolds numbers greater than 10^5, the length scales become very small and the turbulence effects cannot be ignored. Therefore, the present analysis is valid for Reynolds numbers of 100–100,000. For a given frequency and ψ_0, the solution of Eqs. (9) is subject to the initial conditions of $\theta(0) = \theta_0$, $E_f(0) = E_{f0}$, and $E_s(0) = E_{s0}$.

The solution of Eqs. (9) is dependent on I_{fs}, which, in turn, is dependent on the phase-difference ψ_0, as well as on the downstream variation in the phase angles, ψ_s and ψ_f. The equations governing the nonlinear spatial development of the phase angles are obtained as follows. For the subharmonic, the momentum equation (4) of the fluctuation is multiplied by $F'^*(x)u'^*(r)\exp(i\omega t)$ and by $F'(x)u'(r)\exp(-i\omega t)$, and the two resulting equations are subtracted. After taking the time average of the resulting difference equation and applying the boundary-layer-type approximations used in deriving Eqs. (9), the following equation is obtained after integrating over r:

$$I_{cs}\frac{d\psi_s}{dx} = \pi S_s + G(S_s,\theta) + \sqrt{E_f}\,|I_w|\sin(2\psi_s - \psi_f - \psi_0 + \phi)$$

where

$$G = \mathrm{Im}\left\{\int_0^\infty \hat{u}'^*\hat{v}'\,\frac{\partial \bar{u}}{\partial r}\,r\mathrm{d}r\right\}$$

$$\phi = \tan^{-1}\{\mathrm{Im}(I_w)/Re(I_w)\} \tag{10a}$$

Equation (10a) describes the nonlinear variation in the phase angle of the subharmonic. The corresponding equation for the fundamental is similarly obtained in the form:

$$I_{cf}\frac{d\psi_s}{dx} = \pi S_f + G(S_f,\theta) - \frac{E_s}{\sqrt{E_f}}\,|I_w|\sin(2\psi_s - \psi_f - \psi_0 + \phi) \tag{10b}$$

The solution of Eqs. (10) is thus subject to the initial value of the phase-difference ψ_0, and is coupled to Eqs. (9) through $E_s(x)$ and $E_f(x)$.

Results and Discussion

Equations (9) show that the effective fundamental-subharmonic interaction term is dependent on ψ. For $\psi = 0$ or π, only the real part of I_w contributes to the interaction process, and the sign reverses between $\psi = 0$ and π. For $\psi = \pi/2$ or $3\pi/2$, only the imaginary part of I_w contributes to the fundamental-subharmonic interaction process, with the sign reversed between $\psi = \pi/2$ and $3\pi/2$. Since ψ is dependent on the initial phase-difference ψ_0, the solution of Eqs. (9) and (10) is, in turn, dependent on ψ_0. To study the effect of ψ_0 on the vortex-pairing process, Eqs. (9) and (10) are solved for several Strouhal numbers at $\psi_0 = 0$, $\pi/2$, π, and $3\pi/2$. The initial conditions for the solution of Eqs. (9) and (10) are taken as $\theta_0 = 0.006$, $E_{f0} = 10^{-6}$, and $E_{s0} = 10^{-8}$, which correspond to excitation velocities of 0.1% u_e and 0.01% u_e, respectively.

For $S_f < 0.4$, the fundamental-subharmonic interaction process was found to be insignificant, as Fig. 1 indicates for

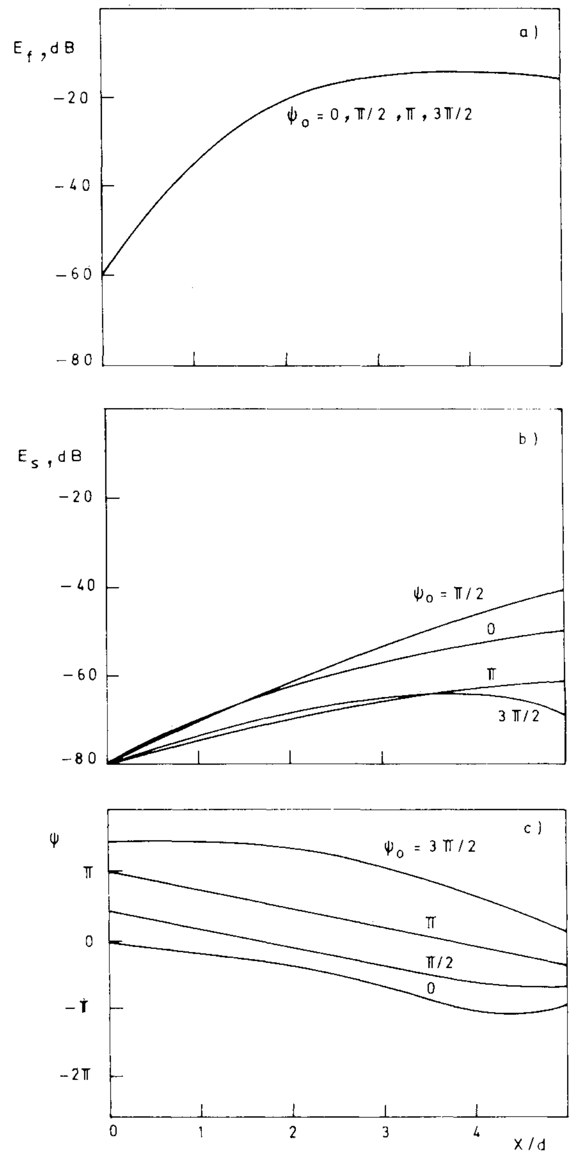

Fig. 2 Effect of ψ_0 on the energies of the fundamental and subharmonic and on their effective phase-difference at $S_f = 0.6$: a) fundamental's energy; b) subharmonic's energy; c) effective phase-difference.

$S_f = 0.2$ ($S_s = 0.1$). For Strouhal numbers $S_f = 0.6$, 0.8, 1.0, and 2.4, the energies of the fundamental and subharmonic and the effective phase-difference ψ are shown in Figs. 2–5, respectively. The figures show that ψ_0 has a pronounced effect on the interaction process. Figures 2a–5a indicate that the growth of the fundamental is independent of ψ_0, which is consistent with the observations of Zhang et al.[3] for a two-dimensional shear layer. During the decay stage of the fundamental, ψ_0 has a pronounced effect on accelerating the decay of the fundamental. This damping effect is proportional to the subharmonic's amplification and, therefore, is a result of the extraction of the fundamental's energy for the growth of the subharmonic.

Figures 2b–5b show that the initial phase-difference has a significant effect on the growth of the subharmonic. Close to the nozzle exit, the subharmonic's initial growth rate is enhanced if ψ_0 is zero, i.e., if both waves are in-phase, and is minimum if ψ_0 is close to π, i.e., out of phase, for all Strouhal numbers considered. However, the subsequent downstream effect of ψ_0 on the development of the subharmonic is dependent on the Strouhal number. An initially higher growth rate at $\psi_0 = 0$ will result in a higher level of the subharmonic close to the nozzle exit. However, this will lead

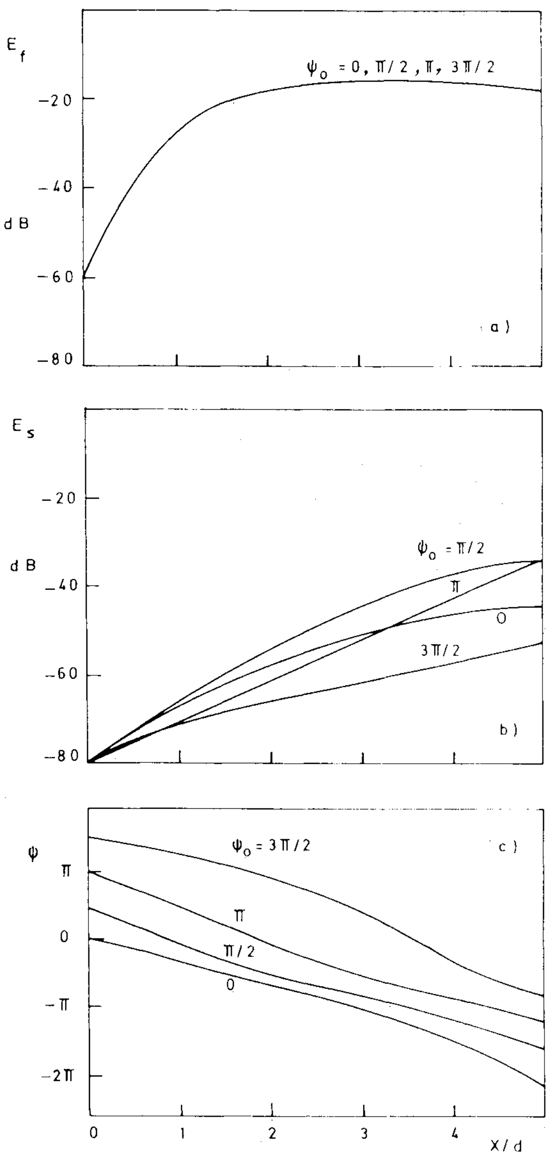

Fig. 3 Effect of ψ_0 on the energies of the fundamental and subharmonic and on their effective phase-difference at $S_f = 0.8$: a) fundamental's energy; b) subharmonic's energy; c) effective phase-difference.

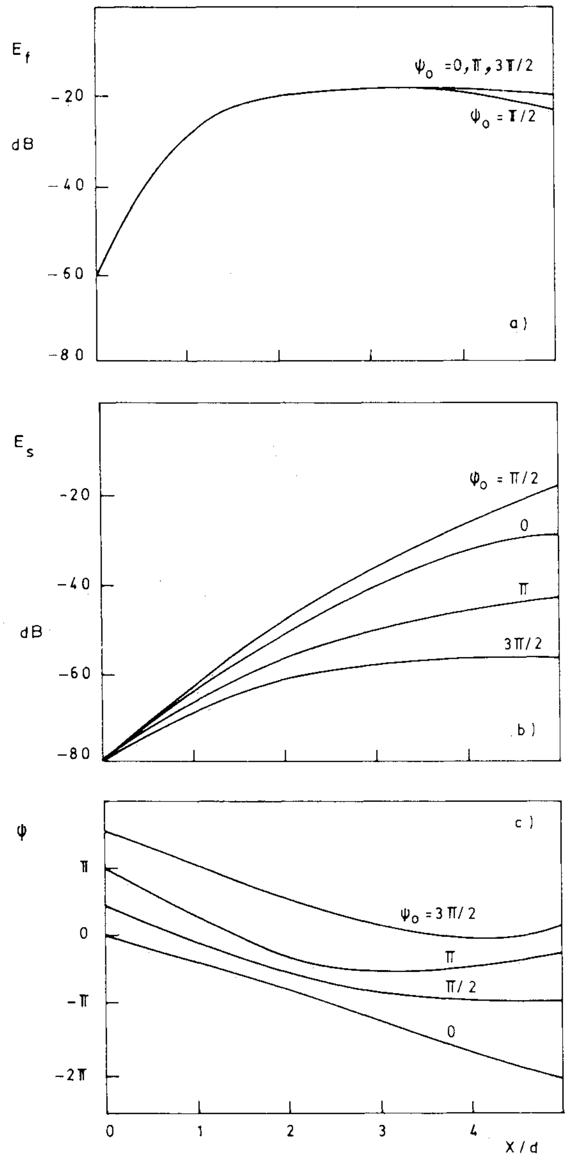

Fig. 4 Effect of ψ_0 on the energies of the fundamental and subharmonic and on their effective phase-difference at $S_f = 1.0$: a) fundamental's energy; b) subharmonic's energy; c) effective phase-difference.

to enhancing the mean-flow energy drain for the initial growth of the subharmonic. Consequently, less mean-flow energy is available for the subsequent downstream growth of the subharmonic along the jet. Therefore, although the initial growth rate of the subharmonic is maximum when the two waves are in-phase, the subsequent peak of the subharmonic can be maximum at other values of ψ_0, depending on Strouhal number.

The initial region of a round jet close to the nozzle exit is similar to the two-dimensional shear-layer case. Therefore, some of the features obtained in Figs. 2–5 can be qualitatively compared to those of the two-dimensional shear layer. Zhang et al.[3] forced a mixing layer at both the fundamental and subharmonic frequencies. The measured subharmonic's initial growth rates were found to decrease by as much as 30% when ψ_0 was varied between 0 and π, see Figs. 2b–5b. This effect can also be seen in the numerical results of Patnaik et al.[1] and Riley and Metcalfe[2] for the stratified mixing layer case. Using the integral approach, Nikotopoulos and Liu[11] have also shown that, for a plane shear layer, the larger the phase-difference, the smaller the subharmonic peak.

In the present model, the mixing rate of the jet is governed by the mean-flow energy drains of the fundamental and subharmonic. The momentum thickness θ is taken here as the measure of the jet's spreading and mixing. The development of θ is governed by Eq. (9a), which shows that, with $I_{cm} < 0$, the jet spreads as long as energy is drained for the growth of the fundamental or the subharmonic. Thus, the growth rate is proportional to E_f and E_s, while I_{pf} and I_{ps} represent the efficiency of each component in extracting energy from the mean flow. The development of the momentum thickness along the jet is shown in Figs. 6–10 for $S_f = 0.2$, 0.6, 0.8, 1.0, and 2.4, respectively. In the initial region of the jet, the level of the fundamental is much higher than that of the subharmonic. Therefore, the initial growth of the jet is governed by the amplification of the fundamental. The growth rate initially increases as E_f increases, and then levels off as I_{pf} decreases. I_{pf} is calculated here based on the linear stability theory in which the amplification rate decreases as the momentum thickness increases. At some downstream stations, the amplification rate is negative corresponding to a damped solution and, hence, I_{pf} is negative. Therefore, after an initial growth region, the growth rate is almost zero where the fundamental decays and the subharmonic amplifies. As the level of the subharmonic becomes large, its role in increasing the spreading rate becomes significant, as indicated in Figs. 9 and 10. Since the growth of the subharmonic is dependent on the initial phase-difference ψ_0, the spreading rate is in turn dependent on ψ_0. The higher the level of the subharmonic the higher the spreading rate.

In order to compare the predicted effect of ψ_0 on the spreading rate with experimental observations, data where the initial phase-difference is controlled should be used. However, in the currently available experimental data for a circular jet under excitation, only the fundamental is controlled. Therefore, ψ_0 is an arbitrary function of the experimental facility. Under such conditions, one can assume that the measured data corresponds to the optimum ψ_0, which produces maximum amplification of the subharmonic. The predicted growth of the jet at maximum amplification of the subharmonic at $S_f = 2.4$ is shown in Fig. 11 and compared with the data of Ref. 12. The abscissa is normalized by λ_0, where λ_0 is the initial wavelength of the fundamental equal to Cd/S_f, and C is the initial nondimensional convection velocity taken to be equal to 0.5, as in the experiment.

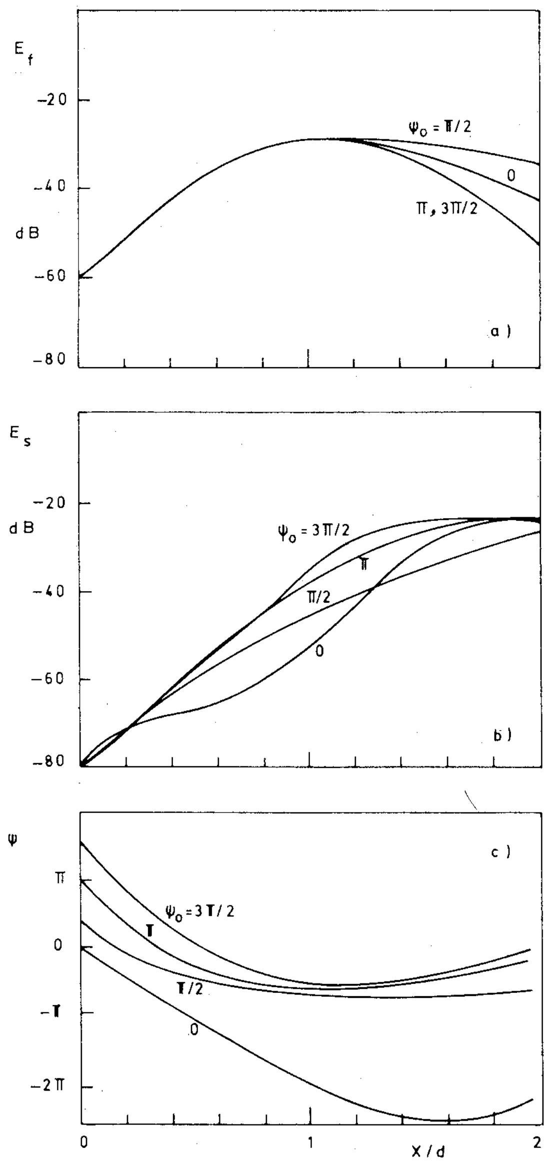

Fig. 5 Effect of ψ_0 on the energies of the fundamental and subharmonic and on their effective phase-difference at $S_f = 2.4$: a) fundamental's energy; b) subharmonic's energy; c) effective phase-difference.

The figure shows qualitative agreement between theory and observations. The growth rate can be divided into several stages: an initial growth stage corresponding to the amplification of the subharmonic, a constant θ stage where the fundamental is decaying, follwed by a second growth stage where the subharmonic is amplifying. Reynolds and Bouchard's[13] measurements of the volume flux of a circular jet under excitation at $S_f = 2.1$ have shown the same qualitative behavior as that of the momentum thickness shown in Fig. 11. This stepwise growth behavior has been discussed in detail in Ref. 6 for the two-dimensional shear-layer case.

In the initial region of the jet, the level of the subharmonic can be considered small with respect to that of the fun-

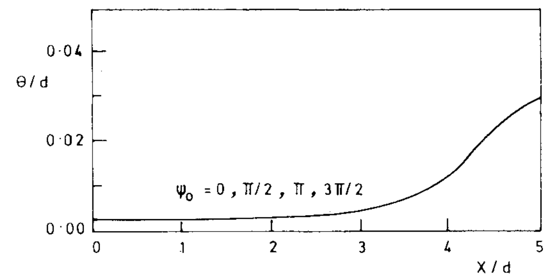

Fig. 6 Development of momentum thickness at $S_f = 0.2$.

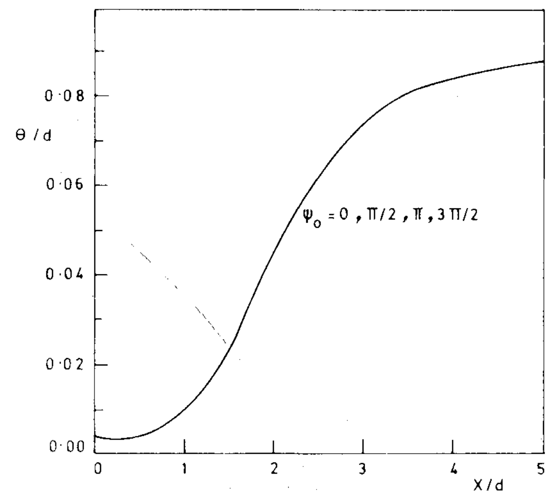

Fig. 7 Development of momentum thickness at $S_f = 0.4$.

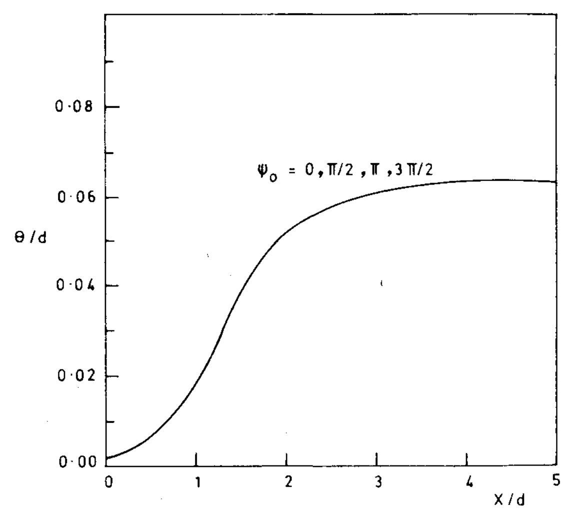

Fig. 8 Development of momentum thickness at $S_f = 0.8$.

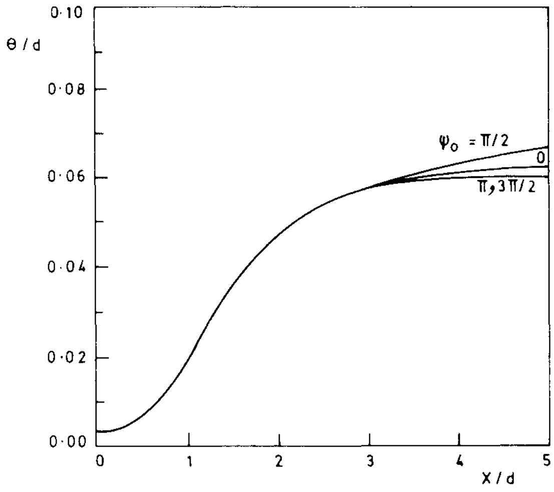

Fig. 9 Development of momentum thickness at $S_f=1.0$.

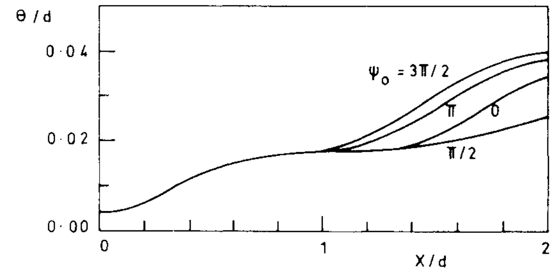

Fig. 10 Development of momentum thickness at $S_f=2.4$.

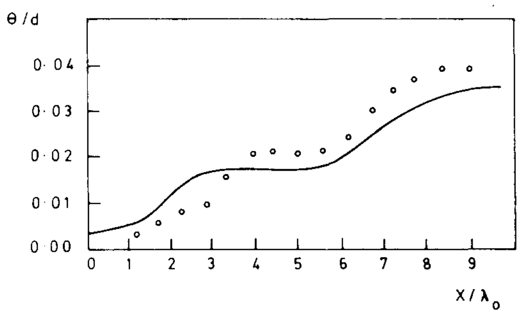

Fig. 11 Comparison of predicted momentum thickness and Laufer and Zhang's measurement.[12]

damental. Hence, the initial growth of the fundamental is independent of the subharmonic as shown in Figs. 1-5. In this case, the second term on the right-hand side of Eq. (9b) can be neglected. In order to examine the mechanisms governing the amplification of the subharmonic, the nonparallel flow effects are neglected for the moment. Hence, the integrals involved in Eqs. (9) and (10) are considered to be independent of θ. Under such approximations, $\psi=\psi_0$ and the fundamental and subharmonic components are given by

$$E_f=E_{f0}\exp(I_{pf}x/I_{cf})$$

$$E_s=E_{s0}\exp(I_{ps}x/I_{cs})\exp[a_1\exp(a_2x)]$$

where

$$a_1=2(I_{fs}/I_{cs})(I_{cf}/I_{pf})\sqrt{E_{f0}}$$

and

$$a_2=\tfrac{1}{2}(I_{pf}/I_{cs}) \qquad (11)$$

Thus, the growth of the subharmonic is due to two instability mechanisms: one arising from the mean-flow profile, and the second arising from the amplified fundamental. Interpreting a spatially periodic mixing layer as the fundamental, Kelly[5] considered the temporal instability of the subharmonic for parallel flow and obtained:

$$\frac{dB}{dt}=CAB^* \qquad (12a)$$

which gives

$$|B(t)|=|B(0)|\exp(Mt) \qquad (12b)$$

where $B=E_s^{1/2}$ and $A=E_f^{1/2}$ in the present nomenclature. Monkewitz[4] extended Kelly's[5] analysis to include an arbitrary phase-difference between the fundamental and the subharmonic, and concluded that an arbitrary initial phase-difference leads to transients with two angles where the growth rate of the subharmonic is initially increased or decreased depending on ψ_0. For $\psi_0=0$, Monkewitz[4] obtained Kelly's[5] result, however, for $\psi_0=\pi$, he obtained

$$|B(t)|=|B(0)|\exp(-Mt) \qquad (13)$$

If one follows the assumption of Kelly[5] and Monkewitz[4] in considering the growth of the subharmonic resulting from the fundamental alone, i.e., $I_{ps}=0$, and considers the amplitude of the fundamental to be constant, i.e., $I_{pf}=0$, Eq. (11) reduces to

$$E_s=E_{s0}\exp(2I_wE_{f0}^{1/2}x/I_{cs}), \qquad \text{for } \psi_0=0 \qquad (14a)$$

$$E_s=E_{s0}\exp(-2I_wE_{f0}^{1/2}x/I_{cs}), \qquad \text{for } \psi_0 f=\pi \qquad (14b)$$

Equations (14a) and (14b) are equivalent to those of Kelly [Eqs. (12)] and Monkewitz [Eq. (13)], respectively, with x replacing t, and $M=I_wE_{f0}^{1/2}/I_{cs}$.

While Kelly[5] and Monkewitz[4] do not consider the influence of the subharmonic on the fundamental, the present analysis allows the subharmonic to react nonlinearly on the fundamental through the second term in Eq. (10b). In the present analysis, the growth of the fundamental due to the subharmonic is given by

$$dE_f/dx=-I_{fs}E_s\sqrt{E_f}/I_{cf} \qquad (15)$$

Equation (15) is equivalent to the form obtained by Maslowe,[14] Kelly,[15] and Weissman[16] for stratified shear layers in the temporal case. The present inclusion of the subharmonic's effect on the fundamental allows the subharmonic to alter the decay rate of the fundamental, as shown in Figs. 2-5.

Conclusions

Using the nonlinear energy integral technique, the effects of initial phase-difference between fundamental and subharmonic instability waves on their spatial developments and on the spreading rate of a laminar circular jet were investigated. The results presented are for the limiting case of high Reynolds numbers where the viscous terms can be neglected. For all Strouhal numbers considered, the initial amplification rate of the subharmonic was found to be maximum when both waves are out of phase, and minimum when the two are in-phase. However, the effect of this initial phase-difference on the subsequent downstream peak was found to vary with the Strouhal number.

With the role of the initial phase-difference on the growth of the subharmonic now established, one can conclude that the jet instability acts as an amplifier not only with respect to selective frequencies, but also with respect to selective phase-differences between the fundamental and the subharmonic. In natural, uncontrolled conditions, several subharmonic components at a given frequency can exist with several random phase-differences with respect to the fundamental. The mean flow acts as a first amplifier that will amplify or suppress the subharmonic depending on its frequency. The fundamental instability wave associated with the mean-flow profile then acts as a second amplifier that will amplify or suppress the subharmonic depending on its arbitrary phase-difference with respect to the fundamental. For a given frequency, the most amplified subharmonic is thus the one with the proper phase-difference. This can be the cause of the observed jitter in the location or strength of pairing. If the random subharmonic component at the nozzle exit is at the proper phase-difference, it will produce strong pairing at some downstream stations. If the phase-difference is not the optimum one, the strength of pairing will be reduced and its location will be altered. Thus, in a circular jet under bimodal excitation, where the initial phase-difference can be controlled, one can manipulate the mixing rate by controlling the initial phase-difference between the two excitation modes.

References

[1]Patnaik, P. C., Sherman, F. S., and Corcos, G. M., "A Numerical Simulation of Kelvin-Helmholtz Waves of Finite Amplitude," *Journal of Fluid Mechanics*, Vol. 73, 1976, pp. 215-240.

[2]Riley, J. J. and Metcalfe, R. W., "Direct Numerical Simulation of Perturbed Mixing Layers," AIAA Paper 80-0274, Jan. 1980.

[3]Zhang, Y. G., Ho, C-M., and Monkewitz, P., "The Mixing Layer Forced by Fundamental and Subharmonic," *Proceedings of IUTAM Symposium on Laminar-Turbulent Transition*, Novosibirsk, USSR, July 1984.

[4]Monkewitz, P. A., "On the Effect of the Phase Difference Between Fundamental and Subharmonic Instability in a Mixing Layer," University of California, Los Angeles, Internal Rept., 1982.

[5]Kelly, R. E., "On the Instability of an Inviscid Shear Layer Which is Periodic in Space and Time," *Journal of Fluid Mechanics*, Vol. 27, 1967, pp. 657-689.

[6]Ho, C-M. and Huang, L. S., "Subharmonics and Vortex Merging in Mixing Layers," *Journal of Fluid Mechanics*, Vol. 119, 1982, pp. 443-473.

[7]Michalke, A., "Instabilitat eines Kompressiblen runden Freistrahls Unter Berücksichtigun des Einflusses der Strahlgrenzschichtdicke," *Zeitschrift fur flugwissenscnaften*, Vol. 19, 1971, pp. 319-328.

[8]Ko, D. R. S., Kubota, T., and Less, L., "Finite Disturbance Effect in the Stability of Laminar Incompressible Wakes Behind a Flate Plate," *Journal of Fluid Mechanics*, Vol. 40, 1970, pp. 315-341.

[9]Stuart, J. T., "Hydrodynamic Stability," *Applied Mechanics Reviews*, Vol. 18, 1965, pp. 523-531.

[10]Morris, P. J., "The Spatial Instability of Axisymmetric Jets," *Journal of Fluid Mechanics*, Vol. 77, 1976, pp. 511-529.

[11]Nikotopoulos, D. and Liu, J. T. C., "Mode Interactions in Developing Shear Flows," *Bulletin of the American Physical Society*, Vol. 27, No. 9, 1982, p. 1192.

[12]Laufer, J. and Zhang, J. P., "Unsteady Aspects of Low Mach Number Jet," *The Physics of Fluids*, Vol. 26, 1983, pp. 1740-1750.

[13]Reynolds, W. C. and Bouchard, E. E., "The Effect of Forcing on the Mixing Layer Region of a Round Jet," *Unsteady Turbulent Shear Flows*, edited by R. Michel, J. Cousteix, and R. Houdeville, 1981, Springer, New York, pp. 401-411.

[14]Maslowe, S. A., "Weakly Nonlinear Stability Theory of Stratified Shear Flows," *Quarterly Journal of the Royal Meteorological Society*, Vol. 103, 1977, pp. 769-783.

[15]Kelly, R. E., "On the Resonant Interaction of Neutral Disturbances in Two-Dimensional Inviscid Shear Flows," *Journal of Fluid Mechanics*, Vol. 31, 1968, pp. 789-799.

[16]Weissmean, M. A., "Nonlinear Wave Packets in the Kelvin-Helmholtz Instability," *Philosophical Transactions of the Royal Society of London,* Ser. A, Vol. 290, 1979, pp. 639-689.